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Abstract A state-based peridynamic material model describes internal forces acting on a point in terms of the collective deformation of all the material within a neighborhood of the point. In this paper, the response of a state-based peridynamic material is investigated for a small deformation superposed on a large deformation. The appropriate notion of a small deformation restricts the relative displacement between points, but it does not involve the deformation gradient (which would be undefined on a crack). The material properties that govern the linearized material response are expressed in terms of a new quantity called the modulus state. This determines the force in each bond resulting from an incremental deformation of itself or of other bonds. Conditions are derived for a linearized material model to be elastic, objective, and to satisfy balance of angular momentum. If the material is elastic, then the modulus state is obtainable from the second Fréchet derivative of the strain energy density function. The equation of equilibrium with a linearized material model is a linear Fredholm integral equation of the second kind. An analogue of Poincaré's theorem is proved that applies to the infinite dimensional space of all peridynamic vector states, providing a condition similar to irrotationality in vector calculus.

Keywords Peridynamics · Linearization · Elasticity · Continuum mechanics · Nonlocal · Constitutive modeling

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1 Introduction

The peridynamic model is an extension of solid mechanics in which a point in a continuum interacts directly with other points separated from it by a finite distance. The maximum

Dedicated to the memory of James K. Knowles.

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interaction distance provides a length scale for a material model, although the model may additionally contain smaller length scales. The purpose of the peridynamic approach is to allow discontinuous media, including continuous media in which discontinuities appear spontaneously as a result of deformation, to be treated according to exactly the same equations as continuous bodies.

Restricting attention to elastic materials, the central assumption in the peridynamic model is that the strain energy density $W(\mathbf{x})$ at a point $\mathbf{x}$ depends collectively on the deformation of all the points in a neighborhood $\mathbf{x}$ with radius $\delta > 0$. δ is called the *horizon* for the material. There is no assumption that the body is continuous within this horizon, *i.e.,* that density is a continuous function of position in the reference configuration; nor is there an assumption that an initially continuous body remains continuous. To keep track of the interactions between $\mathbf{x}$ and all of its neighbors within its horizon, mappings called *peridynamic states* are used as a convenience. These mappings allow constitutive relations to be expressed in a concise form. These relations prescribe how the collective deformation of the material within the horizon of $\mathbf{x}$ determines the nonlocal forces acting on $\mathbf{x}$.

A general framework for constitutive modeling in terms of states has been presented in [1]. The main purpose of the present paper is to develop a linearized version of the peridynamic state theory applicable to small deformations. The appropriate notion of small deformation is different in the peridynamic theory from the standard theory, because the peridynamic notion allows for discontinuities such as cracks in the deformation. By linearizing the general theory, simplifications are obtained that specialize the peridynamic equation of equilibrium to a Fredholm linear integral equation of the second kind. Conditions are obtained for a linearized material to be elastic and to satisfy objectivity and balance of angular momentum. The material properties in the linearized theory are contained in a quantity called the *modulus state* that is analogous to the fourth order elasticity tensor in the standard theory. For an elastic material, the modulus state is obtained from the second Fréchet derivative of the strain energy density function; this is analogous to the second tensor gradient in the standard theory.

The main advantages of the peridynamic method appear in problems involving discontinuities that are either present initially or evolve as a result of deformation, as in the case of dynamic fracture. Although this paper does not specifically address damage evolution, it is shown that the applicable notion of a "small deformation" that permits linearization (see Sect. 4.1) is compatible with such discontinuities. The linearized equations also permit softening materials to be studied, including the associated material stability issues [2]. The linearized model also provides a setting in which to explore the properties of dispersive linear waves and their stability, a topic which is touched on briefly in example problem 4 below (see Sect. 5.4).

In this paper we summarize in Sect. 2 the basic mathematical tools used in the peridynamic theory. Section 3 contains a discussion of elastic material models in the fully nonlinear theory, including new results on the connection between objectivity and balance of angular momentum. Section 4 presents the linearized version of the theory, including the characterization of a material in terms of the modulus state. Necessary and sufficient conditions for a material to be elastic are proved in terms of the properties of the modulus state. Conditions for objectivity and balance of angular momentum in the linearized theory are proved. The equation of motion is shown to reduce to a linear integro-differential equation in terms of displacement whose coefficients are derivable from the original nonlinear constitutive model. The properties of the linear theory in the limit of small horizon are discussed in Sect. 4.7, where it is shown that under suitable restrictions, the model converges to the standard theory. Examples of linearization are given in Sect. 5. In Sect. 6 is presented a brief comparison between the peridynamic theory, with and without linearization, and the standard theory of solid mechanics. The Appendix contains details about the properties of second Fréchet derivatives of functions of peridynamic states.

2 Peridynamic States

Constitutive modeling within the peridynamic theory considers the collective deformation of all the material within a neighborhood of any point $\mathbf{x} \in \mathcal{B}$, where $\mathcal{B}$ is the reference configuration of the body. To treat the kinematics and kinetics of such a neighborhood, it is convenient to introduce mathematical objects called *peridynamic states* [1].

Let $\delta > 0$ denote the horizon, which is treated as a material property. For any $\mathbf{q} \in \mathcal{B}$ such that $|\mathbf{q} - \mathbf{x}| \leq \delta$, the vector $\boldsymbol{\xi} = \mathbf{q} - \mathbf{x}$ is called a *bond*. The set of all such bonds is called the *family* of $\mathbf{x}$ and denoted $\mathcal{H}$.

A peridynamic state $\underline{A}$ is a mapping that associates with any $\boldsymbol{\xi} \in \mathcal{H}$ some other quantity denoted $\underline{A}\langle\boldsymbol{\xi}\rangle$. (Angle brackets are used to distinguish the bond that a state operates on from dependencies that the state itself may have on other quantities, such as position.) If the value of $\underline{A}\langle\boldsymbol{\xi}\rangle$ is a scalar, then $\underline{A}$ is called a *scalar state*. Most of the states considered in this paper are of either of two types, called *vector states* and *double states*. These will now be defined.

2.1 Vector States

If $\underline{\mathbf{A}}$ is a vector state, then for any $\boldsymbol{\xi} \in \mathcal{H}$, the value of $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$ is a vector in $\mathbb{R}^3$. The set of all vector states is denoted $\mathcal{V}$. The *dot product* of two vector states $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ is defined by

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}$$

where the symbol "$\cdot$" denotes the inner product of two vectors in $\mathbb{R}^3$, *i.e.,*

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{A}_i\langle\boldsymbol{\xi}\rangle \underline{B}_i\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}.$$

Here, $\underline{A}_i$ and $\underline{B}_i$ represent the components of $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ respectively in an orthonormal basis, and the summation convention is used.

The composition of two vector states $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ is defined by

$$(\underline{\mathbf{A}} \circ \underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle = \underline{\mathbf{A}}\langle\underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle\rangle$$

for any bond vector $\boldsymbol{\xi}$. The identity vector state $\underline{\mathbf{X}}$ is defined by

$$\underline{\mathbf{X}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi} \quad \forall\boldsymbol{\xi} \in \mathcal{H}.$$

The norm of a vector state $\underline{\mathbf{A}}$ is defined by

$$\|\underline{\mathbf{A}}\| = \sqrt{\underline{\mathbf{A}} \bullet \underline{\mathbf{A}}}. \tag{1}$$

A vector state $\underline{\mathbf{Q}} \in \mathcal{V}$ is *orthogonal* if there is a proper orthogonal tensor $\mathbf{Q}$ such that for every bond vector $\boldsymbol{\xi}$,

$$\underline{\mathbf{Q}}\langle\boldsymbol{\xi}\rangle = \mathbf{Q}\boldsymbol{\xi}.$$

Geometrically, orthogonal states rigidly rotate the bonds in the family.

2.2 Double States

If $\underline{\mathbb{D}}$ is a *double state*, then the value of $\underline{\mathbb{D}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle$ is a second-order tensor, where $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are bonds in $\mathcal{H}$. The set of all double states is denoted $\mathcal{D}$. If $\underline{\mathbb{D}}$ is a double state, let the *left product* of $\underline{\mathbf{A}}$ and $\underline{\mathbb{D}}$ be the vector state $\underline{\mathbf{A}}\bullet\underline{\mathbb{D}}$ defined by

$$(\underline{\mathbf{A}}\bullet\underline{\mathbb{D}})_j\langle\boldsymbol{\xi}\rangle=\int_{\mathcal{H}}\underline{A}_i\langle\boldsymbol{\zeta}\rangle\underline{D}_{ij}\langle\boldsymbol{\zeta},\boldsymbol{\xi}\rangle\,dV_{\zeta}\quad\forall\boldsymbol{\xi}\in\mathcal{H}.$$

Similarly, let the *right product* of $\underline{\mathbb{D}}$ and $\underline{\mathbf{B}}$ be defined by

$$(\underline{\mathbb{D}}\bullet\underline{\mathbf{B}})_i\langle\boldsymbol{\xi}\rangle=\int_{\mathcal{H}}\underline{D}_{ij}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle\underline{B}_j\langle\boldsymbol{\zeta}\rangle\,dV_{\zeta}\quad\forall\boldsymbol{\xi}\in\mathcal{H}.$$

If $\underline{\mathbb{E}}$ is also a double state, then $\underline{\mathbb{D}}\bullet\underline{\mathbb{E}}$ is defined by

$$(\underline{\mathbb{D}}\bullet\underline{\mathbb{E}})_{ij}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle=\int_{\mathcal{H}}\underline{D}_{ik}\langle\boldsymbol{\xi},\boldsymbol{\lambda}\rangle\underline{E}_{kj}\langle\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle\,dV_{\lambda}.$$

Denote by $\underline{\mathbb{D}}^{\dagger}$ the *adjoint* of $\underline{\mathbb{D}}$, defined by

$$\underline{D}^{\dagger}_{ij}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle=\underline{D}_{ji}\langle\boldsymbol{\zeta},\boldsymbol{\xi}\rangle\qquad\forall\boldsymbol{\xi},\boldsymbol{\zeta}\in\mathcal{H}.\tag{2}$$

Note that the order of the bonds, as well as the order of the indices, is switched when taking the adjoint. Observe from (2) that for any vector states $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$,

$$\underline{\mathbf{B}}\bullet\underline{\mathbb{D}}^{\dagger}\bullet\underline{\mathbf{A}}=\underline{\mathbf{A}}\bullet\underline{\mathbb{D}}\bullet\underline{\mathbf{B}}.\tag{3}$$

$\underline{\mathbb{D}}$ is *self-adjoint* if

$$\underline{\mathbb{D}}^{\dagger}=\underline{\mathbb{D}}.$$

2.3 Fréchet Derivatives of a Function of a Vector State

Let Ψ be a real valued function on the set of vector states, *i.e.,* $\Psi(\cdot):\mathcal{V}\to\mathbb{R}$. If Ψ is Fréchet differentiable at $\underline{\mathbf{A}}\in\mathcal{V}$, then for any $\underline{\mathbf{a}}\in\mathcal{V}$,

$$\Psi(\underline{\mathbf{A}}+\underline{\mathbf{a}})=\Psi(\underline{\mathbf{A}})+\nabla\Psi(\underline{\mathbf{A}})\bullet\underline{\mathbf{a}}+o(\|\underline{\mathbf{a}}\|)\tag{4}$$

where the Fréchet derivative $\nabla\Psi(\underline{\mathbf{A}})$ is a vector state.

Similarly, if $\underline{\mathbf{S}}$ is vector state valued function on $\mathcal{V}$, *i.e.*, $\underline{\mathbf{S}}(\cdot):\mathcal{V}\to\mathcal{V}$, then

$$\underline{\mathbf{S}}(\underline{\mathbf{A}}+\underline{\mathbf{a}})=\underline{\mathbf{S}}(\underline{\mathbf{A}})+\nabla\underline{\mathbf{S}}(\underline{\mathbf{A}})\bullet\underline{\mathbf{a}}+o(\|\underline{\mathbf{a}}\|)\tag{5}$$

where $\nabla\underline{\mathbf{S}}(\underline{\mathbf{A}})$ is a double state. If $\nabla\Psi$ is Fréchet differentiable, then the second Fréchet derivative of Ψ is a double state defined by

$$\nabla\nabla\Psi=\nabla(\nabla\Psi)\quad\text{on }\mathcal{V}.$$

The following list summarizes, omitting some details, three important results that are proved in the Appendix:

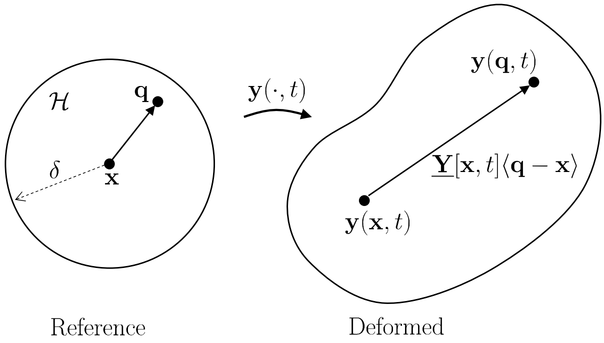

Fig. 1 The deformation state $\underline{\mathbf{Y}}$ maps a bond $\mathbf{q} - \mathbf{x}$ into its deformed image

- If $\Phi(\cdot, \cdot)$ is a function of two vector states, then the order of differentiation in mixed second Fréchet derivatives of Φ is interchangeable.
- If $\Psi(\cdot)$ is a function of a vector state, then $\nabla\nabla\Psi$ is self-adjoint.
- If $\underline{\mathbf{S}}(\cdot)$ is a vector state valued function of a vector state, then $\nabla\underline{\mathbf{S}}$ is self-adjoint if and only if there exists a scalar valued function Ψ such that $\underline{\mathbf{S}} = \nabla\Psi$. (This result is analogous to Poincaré's theorem in vector calculus.)

3 Elastic Peridynamic Materials

This section summarizes some properties of simple materials and elastic materials; details may be found in [1]. Consider the reference configuration of a peridynamic body $\mathcal{B}$, and let $\mathbf{y}$ be a motion of $\mathcal{B}$; thus, the position of a point $\mathbf{x} \in \mathcal{B}$ at time $t \geq 0$ is $\mathbf{y}(\mathbf{x}, t)$. Let $\underline{\mathbf{Y}}[\mathbf{x}, t]$ be the *deformation state*, the vector state defined by

$$\underline{\mathbf{Y}}[\mathbf{x}, t]\langle \mathbf{q} - \mathbf{x} \rangle = \mathbf{y}(\mathbf{q}, t) - \mathbf{y}(\mathbf{x}, t), \quad (\mathbf{q} - \mathbf{x}) \in \mathcal{H} \tag{6}$$

(see Fig. 1). The notation defined in the left hand side of (6) is helpful because we will use $\underline{\mathbf{Y}}$, as opposed to its specific value $\underline{\mathbf{Y}}\langle \mathbf{q} - \mathbf{x} \rangle$ operating on the bond $\mathbf{q} - \mathbf{x}$, as the basic kinematical quantity for purposes of constitutive modeling analogous to the deformation gradient tensor in the standard theory.

The equation of motion is

$$\rho(\mathbf{x})\ddot{\mathbf{y}}(\mathbf{x}, t) = \int_{\mathcal{H}} \left\{ \underline{\mathbf{T}}[\mathbf{x}, t]\langle \mathbf{q} - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{q}, t]\langle \mathbf{x} - \mathbf{q} \rangle \right\} dV_{\mathbf{q}} + \hat{\mathbf{b}}(\mathbf{x}, t) \tag{7}$$

where $\hat{\mathbf{b}}$ is a prescribed body force density field, and where $\underline{\mathbf{T}}$ is a vector state field called the *force state field*. If the material is *simple*, then $\underline{\mathbf{T}}[\mathbf{x}, t]$ is determined from the collective deformation of the family of $\mathbf{x}$ according to the following constitutive model:

$$\underline{\mathbf{T}}[\mathbf{x}, t] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}[\mathbf{x}, t], \mathbf{x}). \tag{8}$$

If, in addition to being simple, the material is also *elastic*, then there exists a function $\hat{W}(\cdot, \cdot) : \mathcal{V} \times \mathbb{R}^3 \to \mathbb{R}$ such that for any $\underline{\mathbf{Y}}$,

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}) = \nabla \hat{W}(\underline{\mathbf{Y}}, \mathbf{x}) \tag{9}$$

in which ∇ denotes the Fréchet derivative with respect to $\underline{\mathbf{Y}}$. The function $\hat{W}$ is called the *strain energy density function*. Elastic materials in the peridynamic theory have many of the same properties as in the classical theory. Among these is the reversible storage of work provided to the body through deformation [1]. The rate of work per unit volume absorbed by a point $\mathbf{x}$ due to interaction with other points in $\mathcal{H}$ is given by

$$\dot{W} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}},$$

which is analogous to the stress power in the classical theory, $\boldsymbol{\sigma} \cdot \dot{\mathbf{F}}$, where $\boldsymbol{\sigma}$ is the Piola stress and $\mathbf{F}$ is the deformation gradient tensor.

A material model $\hat{\underline{\mathbf{T}}}$ is *objective* [1] if, for every orthogonal state $\underline{\mathbf{Q}}$, every $\underline{\mathbf{Y}}$, and every $\mathbf{x}$,

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Q}} \circ \underline{\mathbf{Y}}, \mathbf{x}) = \underline{\mathbf{Q}} \circ \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}). \tag{10}$$

As shown in [1], if the material is elastic, it is objective if and only if

$$\hat{W}(\underline{\mathbf{Q}} \circ \underline{\mathbf{Y}}, \mathbf{x}) = \hat{W}(\underline{\mathbf{Y}}, \mathbf{x}). \tag{11}$$

The strain energy density function of an objective material is therefore invariant with respect to rigid rotations of the family *after* a deformation of the family by $\underline{\mathbf{Y}}$.

3.1 Objectivity and Balance of Angular Momentum

As proved in [1], global balance of angular momentum holds in a body whose material model is such that the following local balance of angular momentum holds for every $\underline{\mathbf{Y}}$ and every $\mathbf{x}$:

$$\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \, dV_{\xi} = \mathbf{0}, \qquad \underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}). \tag{12}$$

The following result is of practical value because it is often easier to show that a material model is objective than to show directly that is satisfies (12).

Proposition 3.1 *Suppose a material model is elastic. Then it is objective if and only if it satisfies the local balance of angular momentum* (12).

Proof Suppose $\hat{W}$ is objective; therefore (11) holds. Deform the family of any $\mathbf{x} \in \mathcal{B}$ (we omit the explicit dependence of $\hat{W}$ on $\mathbf{x}$ to shorten the notation), and let $\underline{\mathbf{Y}}_0$ be the resulting deformation state. Let $\mathbf{c}$ be an arbitrary unit vector. Apply a rigid rotation about $\mathbf{c}$ with rotation angle θ to the deformed family. Let $\underline{\mathbf{Q}}_{\theta}$ be the corresponding proper orthogonal state. Define

$$\underline{\mathbf{Y}}_{\theta} = \underline{\mathbf{Q}}_{\theta} \circ \underline{\mathbf{Y}}_0, \qquad \underline{\mathbf{T}}_{\theta} = \nabla \hat{W}(\underline{\mathbf{Y}}_{\theta}). \tag{13}$$

From the geometry of rigid rotations, for any $\boldsymbol{\xi} \in \mathcal{H}$,

$$\frac{d\underline{\mathbf{Y}}_{\theta}\langle\boldsymbol{\xi}\rangle}{d\theta} = \mathbf{c} \times \underline{\mathbf{Y}}_{\theta}\langle\boldsymbol{\xi}\rangle. \tag{14}$$

By the properties of the Fréchet derivative (4), and using the second of (13) and (14),

$$\frac{d\hat{W}(\underline{\mathbf{Y}}_{\theta})}{d\theta} = \underline{\mathbf{T}}_{\theta} \bullet \frac{d\underline{\mathbf{Y}}_{\theta}}{d\theta}$$

$$= \int_{\mathcal{H}} \underline{\mathbf{T}}_\theta \langle \boldsymbol{\xi} \rangle \cdot (\mathbf{c} \times \underline{\mathbf{Y}}_\theta \langle \boldsymbol{\xi} \rangle) \, dV_{\boldsymbol{\xi}}.$$

Using the vector identity $\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = \mathbf{q} \cdot (\mathbf{r} \times \mathbf{p})$, this implies

$$\frac{d\hat{W}(\underline{\mathbf{Y}}_\theta)}{d\theta} = \mathbf{c} \cdot \int_{\mathcal{H}} \underline{\mathbf{Y}}_\theta \langle \boldsymbol{\xi} \rangle \times \underline{\mathbf{T}}_\theta \langle \boldsymbol{\xi} \rangle \, dV_{\boldsymbol{\xi}}. \tag{15}$$

But since, by assumption, the material is objective, it follows from (11) and (13) that

$$\frac{d\hat{W}(\underline{\mathbf{Y}}_\theta)}{d\theta} = 0. \tag{16}$$

Therefore (15) implies

$$\mathbf{c} \cdot \int_{\mathcal{H}} \underline{\mathbf{Y}}_\theta \langle \boldsymbol{\xi} \rangle \times \underline{\mathbf{T}}_\theta \langle \boldsymbol{\xi} \rangle \, dV_{\boldsymbol{\xi}} = 0$$

for every unit vector $\mathbf{c}$. This can only be true if

$$\int_{\mathcal{H}} \underline{\mathbf{Y}}_\theta \langle \boldsymbol{\xi} \rangle \times \underline{\mathbf{T}}_\theta \langle \boldsymbol{\xi} \rangle \, dV_{\boldsymbol{\xi}} = \mathbf{0} \quad \forall \theta. \tag{17}$$

Setting $\theta = 0$, therefore

$$\int_{\mathcal{H}} \underline{\mathbf{Y}}_0 \langle \boldsymbol{\xi} \rangle \times \underline{\mathbf{T}}_0 \langle \boldsymbol{\xi} \rangle \, dV_{\boldsymbol{\xi}} = \mathbf{0}.$$

Since this must hold for every $\underline{\mathbf{Y}}_0 \in \mathcal{V}$, (12) follows; therefore the balance of angular momentum holds.

Conversely, if (12) holds, then clearly (17) holds. Reversing the above steps, for any unit vector $\mathbf{c}$, (15) and (17) imply (16). Therefore, using (13),

$$\hat{W}(\underline{\mathbf{Q}}_\theta \circ \underline{\mathbf{Y}}_0) - \hat{W}(\underline{\mathbf{Y}}_0) = \int_0^\theta \frac{d\hat{W}(\underline{\mathbf{Y}}_\tau)}{d\tau} \, d\tau = 0 \quad \forall \theta.$$

Thus (11) holds, and the material is therefore objective. □

4 Linearized Peridynamic Models

Linearization of the *bond-based* peridynamic theory was discussed in [3]. In the bond-based theory, each bond has its own constitutive model and responds independently of the other bonds. As shown in [1], the bond-based version is a special case of the *state-based* theory, which allows for collective response of all the bonds in the family. The bond-based description suffers from severe limitations on the types of materials that can be accurately described, notably the restriction on the Poisson ratio $\nu = 1/4$ for isotropic solids. Several studies have investigated the linearized bond-based theory, including [4–9].

In the present paper, linearization within the more general case of state-based models is investigated. This version has far fewer restrictions on material response than the earlier approach and allows any Poisson ratio.

4.1 Small Deformations

Let $\mathcal{B}$ be a closed, bounded body. Consider a deformation $\mathbf{y}^0$, which may be large. Let $\mathbf{u}$ be a displacement field superposed on $\mathbf{y}^0$. The *displacement state field* associated with $\mathbf{u}$ is defined by

$$\underline{\mathbf{U}}[\mathbf{x},t]\langle\mathbf{q}-\mathbf{x}\rangle = \mathbf{u}(\mathbf{q},t) - \mathbf{u}(\mathbf{x},t), \quad \forall \mathbf{x}\in\mathcal{B}, (\mathbf{q}-\mathbf{x})\in\mathcal{H}. \tag{18}$$

To talk about linearization, it is first necessary to introduce a notion of smallness to help establish the conditions under which the linearized version is expected to be applicable. This motivates the following definition:

Definition 4.1 Let $\mathcal{B}$ be a body with horizon δ. A displacement field $\mathbf{u}$ on $\mathcal{B}$ is *small* if, for any $t \geq 0$,

$$\ell \ll \delta \tag{19}$$

where

$$\ell = \sup_{|\mathbf{q}-\mathbf{x}|\leq\delta} |\mathbf{u}(\mathbf{q},t) - \mathbf{u}(\mathbf{x},t)|. \tag{20}$$

This definition is a nonlocal analogue of the standard assumption in the classical theory of linear elasticity that $|\text{grad}\,\mathbf{u}| \ll 1$. Definition 4.1 does not restrict rigid translations of a body, but it does restrict rigid body rotations to small angles. Also, it allows for possible small discontinuities in $\mathbf{u}$.

Also note from (1) and (20) that if ℓ exists, then, for any $\mathbf{x}\in\mathcal{B}$,

$$\begin{aligned}\|\underline{\mathbf{U}}[\mathbf{x},t]\| &= \left[\int_{\mathcal{H}} (\underline{\mathbf{U}}[\mathbf{x},t]\langle\boldsymbol{\xi}\rangle)\cdot(\underline{\mathbf{U}}[\mathbf{x},t]\langle\boldsymbol{\xi}\rangle)\,dV_{\boldsymbol{\xi}}\right]^{1/2} \\ &\leq \left[\int_{\mathcal{H}} \ell^2\,dV_{\boldsymbol{\xi}}\right]^{1/2} = \ell\sqrt{\text{vol}\,\mathcal{H}}.\end{aligned}$$

Therefore, for a small deformation, we can write

$$\|\underline{\mathbf{U}}\| = O(\ell). \tag{21}$$

4.2 Linearization of an Elastic Constitutive Model

Let $\mathcal{B}$ be a closed, bounded body composed of a simple material, with constitutive model given by (8). Consider an equilibrated deformation $\mathbf{y}^0$ corresponding to a time-independent external body force density field $\mathbf{b}^0$. Let $\mathbf{u}$ be a small displacement field superposed on $\mathbf{y}^0$. Linearizing the function $\hat{\underline{\mathbf{T}}}$ given by (8) near $\underline{\mathbf{Y}}^0$ leads to

$$\underline{\mathbf{T}}(\underline{\mathbf{U}}) = \underline{\mathbf{T}}^0 + \underline{\underline{\mathbb{K}}} \bullet \underline{\mathbf{U}} \tag{22}$$

where

$$\underline{\mathbf{T}}^0 = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}^0),$$

$$\underline{\mathbf{Y}}^0[\mathbf{x}]\langle\mathbf{q}-\mathbf{x}\rangle = \mathbf{y}^0(\mathbf{q}) - \mathbf{y}^0(\mathbf{x}),$$

$\underline{\mathbf{U}}$ is defined in (18), and $\underline{\underline{\mathbb{K}}}$ is a double state called the *modulus state* defined by

$$\underline{\underline{\mathbb{K}}} = \nabla\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}^0). \tag{23}$$

As before, the symbol ∇ indicates the Fréchet derivative with respect to $\underline{\mathbf{Y}}$. In this section, heterogeneity in the body is allowed but not explicitly included in the notation; thus we write $\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})$ instead of $\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$.

To evaluate the accuracy with which the linearized model approximates the full model, from (5), (21), (22), and (23), it follows that

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}^0 + \underline{\mathbf{U}}) = \underline{\mathbf{T}}(\underline{\mathbf{U}}) + o(\ell). \tag{24}$$

So, in this sense, the linearized model for a simple material is a first order approximation for small deformations. Note that $\underline{\underline{\mathbb{K}}}$ is independent of t and of $\underline{\mathbf{U}}$. It can depend on $\mathbf{x}$ because $\hat{\underline{\mathbf{T}}}$ can depend explicitly on $\mathbf{x}$ in a heterogeneous body.

If the material is elastic, then the linearized force state given by (22) is obtainable from

$$\underline{\mathbf{T}} = \nabla W$$

where the function W is defined by

$$W(\underline{\mathbf{U}}) = \hat{W}(\underline{\mathbf{Y}}^0) + \underline{\mathbf{T}}^0 \bullet \underline{\mathbf{U}} + \frac{1}{2}\underline{\mathbf{U}} \bullet \underline{\underline{\mathbb{K}}} \bullet \underline{\mathbf{U}},$$

as is easily confirmed by evaluating the Fréchet derivative of this W. This W is an approximation to $\hat{W}$ near a specific $\underline{\mathbf{Y}}^0$.

If the material is elastic, then from (9) and (23),

$$\underline{\underline{\mathbb{K}}} = \nabla\nabla\hat{W}(\underline{\mathbf{Y}}^0),$$

and, by Lemma A.3,

$$\underline{\underline{\mathbb{K}}}^\dagger = \underline{\underline{\mathbb{K}}}.$$

The following is a stronger result that provides a necessary and sufficient condition for a material to be elastic in terms of the modulus state evaluated at arbitrary large deformations.

Proposition 4.1 *Let a constitutive model for a material be given by the continuously Fréchet differentiable function* $\hat{\underline{\mathbf{T}}}(\cdot) : \mathcal{V} \to \mathcal{V}$, *and define a function* $\hat{\underline{\underline{\mathbb{K}}}}(\cdot) : \mathcal{V} \to \mathcal{D}$ *by*

$$\hat{\underline{\underline{\mathbb{K}}}}(\underline{\mathbf{Y}}) = \nabla\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}) \quad \forall\, \underline{\mathbf{Y}} \in \mathcal{V}.$$

Then the material is elastic if and only if

$$\hat{\underline{\underline{\mathbb{K}}}}^\dagger = \hat{\underline{\underline{\mathbb{K}}}} \quad \text{on } \mathcal{V}. \tag{25}$$

Proof The result follows immediately from Lemma A.4. □

4.3 Objectivity and Balance of Angular Momentum in Linearized Models

This section proposes a definition for objectivity in a linearized material model derived from objectivity in the nonlinear theory, (10). To arrive at this, set $\underline{\mathbf{Y}} = \underline{\mathbf{Y}}^0$ and let $\underline{\mathbf{Q}}$ correspond to a rigid rotation through a small angle θ about a unit vector $\mathbf{c}$. Let $\mathbf{Q}$ be the corresponding proper orthogonal tensor, therefore

$$\mathbf{Q} = \mathbf{1} + \theta \mathbf{W} + O(\theta^2), \qquad W_{ij} = \epsilon_{ikj} c_k$$

where ϵ_{ikj} is the alternator symbol. This relation together with (10) and (22) implies

$$\underline{\mathbf{T}}^0 + \underline{\mathbb{K}} \bullet (\theta \mathbf{W} \underline{\mathbf{Y}}^0) = (\mathbf{1} + \theta \mathbf{W}) \underline{\mathbf{T}}^0 + O(\theta^2).$$

Retaining only first order terms in θ motivates the following definition:

Definition 4.2 A linearized material model (22) is *linearly objective* if for every unit vector $\mathbf{c}$,

$$\underline{\mathbb{K}} \bullet (\mathbf{W} \underline{\mathbf{Y}}^0) - \mathbf{W} \underline{\mathbf{T}}^0 = \mathbf{0}$$

where $\mathbf{W}$ is given by

$$W_{ij} = \epsilon_{ikj} c_k.$$

Writing out these last two expressions yields

$$\begin{aligned} 0 &= \int_{\mathcal{H}} \underline{K}_{ij} \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle \, W_{jm} \, \underline{Y}^0_m \langle \boldsymbol{\zeta} \rangle \, dV_{\zeta} - W_{im} \underline{T}^0_m \langle \boldsymbol{\xi} \rangle \\ &= \int_{\mathcal{H}} \underline{K}_{ij} \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle \, \epsilon_{jkm} c_k \, \underline{Y}^0_m \langle \boldsymbol{\zeta} \rangle \, dV_{\zeta} - \epsilon_{ikm} c_k \underline{T}^0_m \langle \boldsymbol{\xi} \rangle \\ &= \left(\int_{\mathcal{H}} \underline{K}_{ij} \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle \, \epsilon_{jkm} \, \underline{Y}^0_m \langle \boldsymbol{\zeta} \rangle \, dV_{\zeta} - \epsilon_{ikm} \underline{T}^0_m \langle \boldsymbol{\xi} \rangle \right) c_k . \end{aligned}$$

Requiring this to hold for every choice of $\mathbf{c}$ results in the following condition for linear objectivity:

Proposition 4.2 *A linearized material model* (22) *is linearly objective if and only if*

$$\epsilon_{jkm} \int_{\mathcal{H}} \underline{K}_{ij} \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle \, \underline{Y}^0_m \langle \boldsymbol{\zeta} \rangle \, dV_{\zeta} - \epsilon_{ikm} \underline{T}^0_m \langle \boldsymbol{\xi} \rangle = 0 \quad \forall \boldsymbol{\xi} \in \mathcal{H}. \tag{26}$$

To obtain a definition of balance of angular momentum for linearized material models, assume that $\hat{\underline{\mathbf{T}}}$ satisfies (12), thus using (22) and (24),

$$\int_{\mathcal{H}} \big(\underline{\mathbf{Y}}^0 \langle \boldsymbol{\xi} \rangle + \underline{\mathbf{U}} \langle \boldsymbol{\xi} \rangle \big) \times \big(\underline{\mathbf{T}}^0 \langle \boldsymbol{\xi} \rangle + (\underline{\mathbb{K}} \bullet \underline{\mathbf{U}}) \langle \boldsymbol{\xi} \rangle + o(\ell) \big) \, dV_{\xi} = \mathbf{0}. \tag{27}$$

By (12), $\int \underline{\mathbf{Y}}^0 \langle \boldsymbol{\xi} \rangle \times \underline{\mathbf{T}}^0 \langle \boldsymbol{\xi} \rangle \, dV_{\xi} = \mathbf{0}$. Since $\underline{\mathbf{U}} = O(\ell)$, requiring (27) to hold to first order in ℓ leads to the following notion of balance of angular momentum appropriate for linearized models.

Definition 4.3 A linearized material model of the form (22) satisfies the *linear balance of angular momentum* if the following condition holds:

$$\int_{\mathcal{H}} \Big(\underline{\mathbf{Y}}^0\langle\boldsymbol{\xi}\rangle \times (\underline{\mathbb{K}} \bullet \underline{\mathbf{U}})\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{U}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{T}}^0\langle\boldsymbol{\xi}\rangle \Big)\, dV_{\boldsymbol{\xi}} = \mathbf{0} \tag{28}$$

for all vector states $\underline{\mathbf{U}}$.

The first term in the integrand in (28) accounts for changes in the bond forces due to $\underline{\mathbf{U}}$. The second term accounts for rotations due to $\underline{\mathbf{U}}$ of bonds with pre-existing forces $\underline{\mathbf{T}}^0\langle\boldsymbol{\xi}\rangle$. These small rotations have a first order effect on the couple $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{T}}^0\langle\boldsymbol{\xi}\rangle$.

If the linearized material model is elastic, then, as in the full nonlinear theory (see Proposition 3.1), there is a close connection between objectivity and angular momentum balance:

Proposition 4.3 *Suppose a linearized material model of the form* (22) *is elastic. Then it satisfies linear objectivity if and only if it satisfies the linear balance of angular momentum.*

Proof Suppose the linear balance of angular momentum (28) holds. The component form of this is

$$\epsilon_{kmj} \int_{\mathcal{H}} \int_{\mathcal{H}} \underline{Y}^0_m\langle\boldsymbol{\xi}\rangle \underline{K}_{ji}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle \underline{U}_i\langle\boldsymbol{\zeta}\rangle\, dV_{\boldsymbol{\zeta}}\, dV_{\boldsymbol{\xi}} + \epsilon_{kim} \int_{\mathcal{H}} \underline{U}_i\langle\boldsymbol{\xi}\rangle\, \underline{T}^0_m\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}} = 0.$$

Interchanging the dummy variables of integration $\boldsymbol{\xi} \leftrightarrow \boldsymbol{\zeta}$ in the double integral leads to

$$\int_{\mathcal{H}} \left(\epsilon_{kmj} \int_{\mathcal{H}} \underline{Y}^0_m\langle\boldsymbol{\zeta}\rangle \underline{K}_{ji}\langle\boldsymbol{\zeta},\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\zeta}} + \epsilon_{kim} \underline{T}^0_m\langle\boldsymbol{\xi}\rangle \right) \underline{U}_i\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}} = 0.$$

Since this must hold for every choice of $\underline{\mathbf{U}}$, it follows that

$$\epsilon_{kmj} \int_{\mathcal{H}} \underline{Y}^0_m\langle\boldsymbol{\zeta}\rangle \underline{K}_{ji}\langle\boldsymbol{\zeta},\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\zeta}} + \epsilon_{kim} \underline{T}^0_m\langle\boldsymbol{\xi}\rangle = 0 \quad \forall \boldsymbol{\xi} \in \mathcal{H}.$$

Since the material is elastic, (25) holds, so

$$\epsilon_{kmj} \int_{\mathcal{H}} \underline{Y}^0_m\langle\boldsymbol{\zeta}\rangle \underline{K}_{ij}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle\, dV_{\boldsymbol{\zeta}} + \epsilon_{kim} \underline{T}^0_m\langle\boldsymbol{\xi}\rangle = 0 \quad \forall \boldsymbol{\xi} \in \mathcal{H}.$$

Using the identities

$$\epsilon_{kmj} = \epsilon_{jkm}, \qquad \epsilon_{kim} = -\epsilon_{ikm},$$

(26) is seen to hold, therefore, by Proposition 4.2, the material model satisfies linear objectivity.

Conversely, reversing the above steps shows that (26) implies (28). Therefore, linear objectivity implies the linear balance of angular momentum. □

The condition (25) is analogous to the major symmetry of the elasticity tensor in the standard theory,

$$\mathsf{C}_{ijkl} = \mathsf{C}_{klij},$$

which applies to elastic materials. The condition (26) is analogous to the minor symmetry in the standard theory,

$$\mathsf{C}_{ijkl} = \mathsf{C}_{jikl}$$

which ensures the symmetry of the stress tensor and therefore, by Cauchy's theorem, also ensures balance of angular momentum.

4.4 Equation of Motion

No additional approximations need to be applied to the equation of motion in the linearized theory. Substituting the linearized constitutive models into this equation transforms it into a linear integro-differential equation expressed in terms of displacement. This equation will now be derived.

As before, assume that a body $\mathcal{B}$ is subjected to time-independent body force density field $\mathbf{b}^0$, resulting in an equilibrated[1] deformation $\mathbf{y}^0$. Then subject the body to an additional body force density field $\mathbf{b}$, so

$$\hat{\mathbf{b}} = \mathbf{b}^0 + \mathbf{b}.$$

Let the resulting change in the displacement field be denoted $\mathbf{u}$, thus

$$\mathbf{y} = \mathbf{y}^0 + \mathbf{u}.$$

It is now more convenient to write volume integrals with points such as $\mathbf{p} \in \mathcal{B}$ as the dummy variable of integration rather than bond vectors such as $\boldsymbol{\xi} \in \mathcal{H}$. To simplify the notation, we will adopt the convention that state quantities take on null values for bond vectors outside the family, *i.e.,* if the material horizon is δ, then

$$\underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{p} - \mathbf{x} \rangle = \mathbf{0} \quad \text{whenever} \quad |\mathbf{p} - \mathbf{x}| > \delta.$$

From (7) and (22), we have

$$\begin{aligned} \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \Big\{ & (\underline{\mathbf{T}}^0[\mathbf{x}] + \underline{\mathbb{K}}[\mathbf{x}] \bullet \underline{\mathbf{U}}[\mathbf{x}])\langle \mathbf{p} - \mathbf{x} \rangle \\ & - (\underline{\mathbf{T}}^0[\mathbf{p}] + \underline{\mathbb{K}}[\mathbf{p}] \bullet \underline{\mathbf{U}}[\mathbf{p}])\langle \mathbf{x} - \mathbf{p} \rangle \Big\} \, dV_{\mathbf{p}} + \hat{\mathbf{b}}(\mathbf{x},t). \end{aligned} \tag{29}$$

Since $\mathbf{y}^0$ is equilibrated, from (7),

$$\int_{\mathcal{B}} \Big\{ \underline{\mathbf{T}}^0[\mathbf{x}]\langle \mathbf{p} - \mathbf{x} \rangle - \underline{\mathbf{T}}^0[\mathbf{p}]\langle \mathbf{x} - \mathbf{p} \rangle \Big\} \, dV_{\mathbf{p}} + \mathbf{b}^0(\mathbf{x}) = \mathbf{0}. \tag{30}$$

Subtracting (30) from (29) yields

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \Big\{ (\underline{\mathbb{K}}[\mathbf{x}] \bullet \underline{\mathbf{U}}[\mathbf{x}])\langle \mathbf{p} - \mathbf{x} \rangle - (\underline{\mathbb{K}}[\mathbf{p}] \bullet \underline{\mathbf{U}}[\mathbf{p}])\langle \mathbf{x} - \mathbf{p} \rangle \Big\} \, dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x},t).$$

[1] The assumption that $\mathbf{y}^0$ is equilibrated is not essential but results in the simplification that the linearized material properties become time-independent.

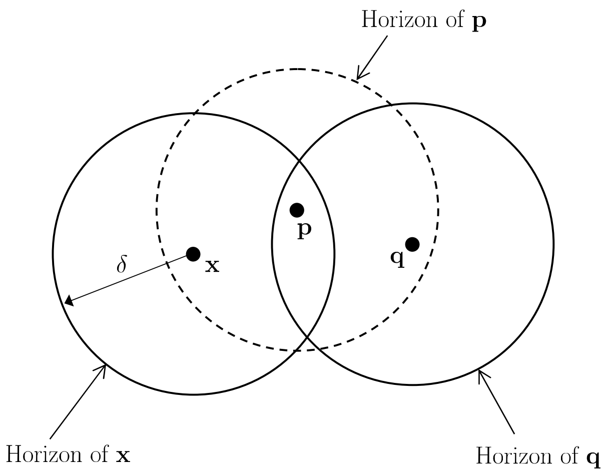

Fig. 2 Point **q** interacts indirectly with **x** even though they are outside each other's horizon because they are both within the horizon of intermediate points such as **p**

Writing out the dot products explicitly using (18) and rearranging leads to

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{p}-\mathbf{x}, \mathbf{q}-\mathbf{x}\rangle (\mathbf{u}(\mathbf{q},t)-\mathbf{u}(\mathbf{x},t))\, dV_{\mathbf{q}}\, dV_{\mathbf{p}} \\ - \int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\mathbb{K}}[\mathbf{p}]\langle \mathbf{x}-\mathbf{p}, \mathbf{q}-\mathbf{p}\rangle (\mathbf{u}(\mathbf{q},t)-\mathbf{u}(\mathbf{p},t))\, dV_{\mathbf{q}}\, dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x},t).$$

After further rearrangement and an interchange of dummy variables of integration $\mathbf{p} \leftrightarrow \mathbf{q}$, this becomes

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \mathbf{C}_0(\mathbf{x},\mathbf{q})\mathbf{u}(\mathbf{q},t)\, dV_{\mathbf{q}} - \mathbf{P}_0(\mathbf{x})\mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) \tag{31}$$

for all $\mathbf{x}$ and t, where $\mathbf{C}_0$ is the tensor valued function defined by

$$\mathbf{C}_0(\mathbf{x},\mathbf{q}) = \int_{\mathcal{B}} \Big(\underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{p}-\mathbf{x}, \mathbf{q}-\mathbf{x}\rangle \\ - \underline{\mathbb{K}}[\mathbf{p}]\langle \mathbf{x}-\mathbf{p}, \mathbf{q}-\mathbf{p}\rangle + \underline{\mathbb{K}}[\mathbf{q}]\langle \mathbf{x}-\mathbf{q}, \mathbf{p}-\mathbf{q}\rangle \Big)\, dV_{\mathbf{p}} \tag{32}$$

and where

$$\mathbf{P}_0(\mathbf{x}) = \int_{\mathcal{B}}\int_{\mathcal{B}} \underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{p}-\mathbf{x}, \mathbf{q}-\mathbf{x}\rangle\, dV_{\mathbf{q}}\, dV_{\mathbf{p}} = \int_{\mathcal{B}} \mathbf{C}_0(\mathbf{x},\mathbf{q})\, dV_{\mathbf{q}}.$$

As illustrated in Fig. 2, $\mathbf{C}_0(\mathbf{x},\mathbf{q})$ may be non-null even though $\delta < |\mathbf{q}-\mathbf{x}| < 2\delta$. This can occur because there are intermediate points **p** whose horizon includes both **x** and **q**. Thus, **x** and **q** can interact indirectly even though they are outside of each other's horizon. This type of indirect interaction appears in the term in the integrand in (32) involving $\underline{\mathbb{K}}[\mathbf{p}]$, since this term arises from the force state at **p**.

In practice, the expressions for $\mathbf{C}_0$ in many materials of interest contain Dirac delta functions centered at **x**. It is convenient to move this term outside the integral in the equation of motion (31) by rewriting it as

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \mathbf{C}(\mathbf{x},\mathbf{q})\mathbf{u}(\mathbf{q},t)\, dV_{\mathbf{q}} - \mathbf{P}(\mathbf{x})\mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t), \tag{33}$$

for all $\mathbf{x}$ and t, where

$$\mathbf{C}(\mathbf{x}, \mathbf{q}) = \mathbf{C}_0(\mathbf{x}, \mathbf{q}) + \boldsymbol{\lambda}(\mathbf{x})\Delta(\mathbf{q} - \mathbf{x}), \tag{34}$$

$$\mathbf{P}(\mathbf{x}) = \mathbf{P}_0(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x}) = \int_{\mathcal{B}} \mathbf{C}(\mathbf{x}, \mathbf{q})\, dV_{\mathbf{q}}, \tag{35}$$

$$\boldsymbol{\lambda}(\mathbf{x}) = -\lim_{\epsilon \to 0} \int_{\mathcal{S}_\epsilon} \mathbf{C}_0(\mathbf{x}, \mathbf{q})\, dV_{\mathbf{q}}, \tag{36}$$

where $\mathcal{S}_\epsilon$ is the interior of a sphere of radius ϵ centered at $\mathbf{x}$, and Δ is the Dirac delta function in $\mathbb{R}^3$. From Proposition 4.1, (32), (34), (35), and (36), the following symmetries hold for any $\mathbf{x}$ and $\mathbf{q}$:

$$\mathbf{C}^T(\mathbf{x}, \mathbf{q}) = \mathbf{C}(\mathbf{q}, \mathbf{x}), \qquad \mathbf{P}^T(\mathbf{x}) = \mathbf{P}(\mathbf{x}).$$

In the special case of a bond-based material description (see Example 1 below), then in addition to this symmetry, one also has

$$\mathbf{C}(\mathbf{x}, \mathbf{q}) = \mathbf{C}(\mathbf{q}, \mathbf{x}).$$

Setting the acceleration term to zero in (33) yields the linearized equation of equilibrium:

$$\int_{\mathcal{B}} \mathbf{C}(\mathbf{x}, \mathbf{q})\mathbf{u}(\mathbf{q})\, dV_{\mathbf{q}} - \mathbf{P}(\mathbf{x})\mathbf{u}(\mathbf{x}) + \mathbf{b}(\mathbf{x}) = \mathbf{0} \tag{37}$$

for all $\mathbf{x}$. This is a Fredholm linear integral equation of the second kind.

4.5 Physical Interpretation of $\mathbf{P}$

The tensor $\mathbf{P}$ has the same mechanical interpretation as in the bond-based theory (see (90) in [3]). To interpret $\mathbf{P}$, consider the deformation with displacement field $\mathbf{v}$ given by

$$\mathbf{v}(\mathbf{x}) = \begin{cases} \mathbf{e} & \text{if } \mathbf{x} = \mathbf{x}_0 \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where $\mathbf{x}_0 \in \mathcal{B}$ is a fixed point and $\mathbf{e}$ is an arbitrary unit vector. The body force density field required to equilibrate this displacement field is found by substituting $\mathbf{v}$ into (37). The integral vanishes, and body force density at $\mathbf{x}_0$ required to sustain the deformation is

$$\mathbf{b}(\mathbf{x}_0) = \mathbf{P}(\mathbf{x}_0)\mathbf{e}.$$

The value of the vector $\mathbf{P}(\mathbf{x}_0)\mathbf{e}$ is therefore the force density (per unit volume) at $\mathbf{x}_0$ required to displace $\mathbf{x}_0$ by the vector $\mathbf{e}$, holding all other points fixed.

4.6 Summary of the Linear Theory

Equation of motion:

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{\mathcal{B}} \mathbf{C}_0(\mathbf{x}, \mathbf{q})\mathbf{u}(\mathbf{q}, t)\, dV_{\mathbf{q}} - \mathbf{P}_0(\mathbf{x})\mathbf{u}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t)$$

where

$$\mathbf{C}_0(\mathbf{x},\mathbf{q}) = \int_{\mathcal{B}} \Big(\underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{p}-\mathbf{x}, \mathbf{q}-\mathbf{x}\rangle - \underline{\mathbb{K}}[\mathbf{p}]\langle \mathbf{x}-\mathbf{p}, \mathbf{q}-\mathbf{p}\rangle + \underline{\mathbb{K}}[\mathbf{q}]\langle \mathbf{x}-\mathbf{q}, \mathbf{p}-\mathbf{q}\rangle \Big)\, dV_{\mathbf{p}},$$

$$\mathbf{P}_0(\mathbf{x}) = \int_{\mathcal{B}} \mathbf{C}_0(\mathbf{x},\mathbf{q})\, dV_{\mathbf{q}}.$$

The linearized material properties are contained in the double state $\underline{\mathbb{K}}$. The material is elastic if and only if $\underline{\mathbb{K}}^{\dagger} = \underline{\mathbb{K}}$, in which case it is related to the strain energy density by

$$\underline{\mathbb{K}} = \nabla\nabla W.$$

Regardless of whether the material is elastic:

- $\underline{\mathbb{K}}$ satisfies (26) $\Longleftrightarrow$ the material is linearly objective.
- $\underline{\mathbb{K}}$ satisfies (28) $\Longleftrightarrow$ the material satisfies linear balance of angular momentum.

If the material is elastic, then (26) and (28) are equivalent.

If $\mathbf{C}_0$ contains terms of the form $\Delta(\mathbf{q}-\mathbf{x})$, it is often convenient to rewrite the equation of motion as described in (33)–(36) and as illustrated in Example 1.

4.7 Limit of Small Horizon

It was shown in [7] and more generally in [10] that if a deformation of a body is classically smooth, then the peridynamic theory converges to the standard theory in the limit $\delta \to 0$. This limiting process is carried out in such a way that the bulk properties of the material at every point (the material's response under homogeneous deformation) are unchanged as δ changes. The limit provides a local description of the internal forces (see (50) of [10]) given by the following Piola stress tensor field:

$$\boldsymbol{\sigma} = \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}}, \qquad \mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \tag{38}$$

where $\hat{\underline{\mathbf{T}}}$ is the constitutive model in the original (nonlocal) peridynamic description.

To investigate the implications of this for the linear peridynamic theory, assume for simplicity that $\underline{\mathbf{T}}^0 = \underline{\mathbf{0}}$ and $\underline{\mathbf{Y}}^0 = \underline{\mathbf{X}}$. From (22), set

$$\hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}})\langle\boldsymbol{\xi}\rangle = (\underline{\mathbb{K}} \bullet \underline{\mathbf{U}})\langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}} \underline{\mathbb{K}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle(\mathbf{H}\boldsymbol{\zeta})\, dV_{\boldsymbol{\zeta}}, \qquad \mathbf{H} = \mathbf{F} - \mathbf{1} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}.$$

From this and (38),

$$\boldsymbol{\sigma} = \int_{\mathcal{H}}\int_{\mathcal{H}} \big(\underline{\mathbb{K}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle(\mathbf{H}\boldsymbol{\zeta})\big) \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\zeta}}\, dV_{\boldsymbol{\xi}},$$

or

$$\sigma_{ij} = \int_{\mathcal{H}}\int_{\mathcal{H}} \big(\underline{K}_{ik}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle(H_{kl}\zeta_l)\big)\xi_j\, dV_{\boldsymbol{\zeta}}\, dV_{\boldsymbol{\xi}}.$$

This may be rewritten as a conventional linear elastic model in the form

$$\sigma_{ij} = \mathsf{C}_{ijkl} H_{kl}, \qquad \mathsf{C}_{ijkl} = \int_{\mathcal{H}}\int_{\mathcal{H}} \underline{K}_{ik}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle \xi_j \zeta_l\, dV_{\boldsymbol{\zeta}}\, dV_{\boldsymbol{\xi}}.$$

Observe that the fourth order elasticity tensor C defined by this equation has the major symmetry $\mathsf{C} = \mathsf{C}^T$ if $\underline{\mathbb{K}}$ is self-adjoint. Therefore, the limiting linear local material model is elastic if the original peridynamic linear material model is elastic.

These results do *not* mean that the linear peridynamic theory is equivalent to the standard theory. They do mean that the standard linear theory is derivable from the peridynamic theory in the sense of a limit, under suitable restrictions on the smoothness of the deformation. In particular, the limit does not exist if cracks or other discontinuities are present. Assuming the smoothness requirements are met, then under this limiting process, the material becomes “more local” because the interaction distance δ becomes smaller. It should be emphasized that the limit of small horizon considered in this section is unrelated to the notion of a small deformation that enables the linear approximations derived earlier in this paper.

5 Examples

These examples illustrate how to obtain $\underline{\mathbb{K}}$ and related quantities from a nonlinear constitutive model and how they result in particular forms of the linearized equation of motion. In all cases, we consider linearization at points in the body far from boundaries and interfaces.

5.1 Example 1: A Bond-Based Material

Consider a homogeneous body composed of an elastic *bond-based material* in which the strain energy density is given by

$$\hat{W}(\underline{\mathbf{Y}}) = \int_{\mathcal{H}} \psi(\underline{e}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\xi})\, dV_{\boldsymbol{\xi}} \tag{39}$$

where ψ is the *bond potential* function and the *extension scalar state* is defined by

$$\underline{e}\langle\boldsymbol{\xi}\rangle = |\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}| \tag{40}$$

for any bond $\boldsymbol{\xi} \in \mathcal{H}$. Note that the first argument of ψ is a scalar, not a scalar *state*. For a small change $d\underline{\mathbf{Y}}$ in the deformation state,

$$d\underline{e}\langle\boldsymbol{\xi}\rangle = \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \cdot d\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}. \tag{41}$$

Hence, applying the chain rule to the integrand in (39),

$$dW = \int_{\mathcal{H}} \psi'(\underline{e}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\xi})(\mathbf{M} \cdot d\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle)\, dV_{\boldsymbol{\xi}}, \quad \mathbf{M} = \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}$$

where ψ' is the first derivative of ψ with respect to its first argument and $\mathbf{M}$ is the deformed bond direction. Then from (4) and (9), the force state for this material is given by

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \psi'(\underline{e}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\xi})\mathbf{M}. \tag{42}$$

We will carry out the linearization of this material model near the reference configuration and assume that the force state is null in the reference configuration, *i.e.*,

$$\psi'(0, \boldsymbol{\xi}) = 0 \quad \text{and} \quad \mathbf{M} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|},$$

thus

$$d\underline{\mathbf{T}} = (d\psi')\mathbf{M} + \psi' d\mathbf{M} = d\psi'\mathbf{M} = (\psi'' d\underline{e})\mathbf{M}.$$

(Linearization near a non-null force state, in which the $d\mathbf{M}$ term must be included, is demonstrated in Example 4 below.) Using (41),

$$\begin{aligned} d\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle &= \psi''(0,\boldsymbol{\xi})(\mathbf{M}\cdot d\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle)\mathbf{M} \\ &= \psi''(0,\boldsymbol{\xi})(\mathbf{M}\otimes\mathbf{M})d\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \\ &= \psi''(0,\boldsymbol{\xi})(\mathbf{M}\otimes\mathbf{M})d\underline{\mathbf{U}}\langle\boldsymbol{\xi}\rangle \\ &= \psi''(0,\boldsymbol{\xi})(\mathbf{M}\otimes\mathbf{M})\int_{\mathcal{H}}\Delta(\boldsymbol{\zeta}-\boldsymbol{\xi})d\underline{\mathbf{U}}\langle\boldsymbol{\zeta}\rangle\, dV_{\zeta} \end{aligned}$$

where Δ denotes the Dirac delta function in $\mathbb{R}^3$. Using (5) and (22), the modulus state is therefore given by

$$\underline{\mathbb{K}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle = \boldsymbol{\gamma}(\boldsymbol{\xi})\Delta(\boldsymbol{\zeta}-\boldsymbol{\xi}) \tag{43}$$

where $\boldsymbol{\gamma}(\boldsymbol{\xi})$ is the tensor given by

$$\boldsymbol{\gamma}(\boldsymbol{\xi}) = \psi''(0,\boldsymbol{\xi})\mathbf{M}\otimes\mathbf{M}.$$

Note that this $\underline{\mathbb{K}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle$ is non-null only if $\boldsymbol{\xi}=\boldsymbol{\zeta}$; mechanically this means that each bond responds independently of the others. From (32)–(36) and (43),

$$\begin{aligned} \mathbf{C}_0(\mathbf{x},\mathbf{q}) &= \int_{\mathcal{B}}\Big\{\boldsymbol{\gamma}(\mathbf{p}-\mathbf{x})\Delta(\mathbf{q}-\mathbf{p}) - \boldsymbol{\gamma}(\mathbf{x}-\mathbf{p})\Delta(\mathbf{q}-\mathbf{x}) + \boldsymbol{\gamma}(\mathbf{x}-\mathbf{q})\Delta(\mathbf{p}-\mathbf{x})\Big\}\, dV_{\mathbf{p}} \\ &= \boldsymbol{\gamma}(\mathbf{q}-\mathbf{x}) + \gamma(\mathbf{x}-\mathbf{q}) - \mathbf{G}\Delta(\mathbf{q}-\mathbf{x}) \end{aligned}$$

where $\mathbf{G}$ is the tensor defined by

$$\mathbf{G} = \int_{\mathcal{H}}\boldsymbol{\gamma}(\boldsymbol{\xi})\, dV_{\xi}.$$

Therefore,

$$\begin{aligned} \mathbf{P}_0(\mathbf{x}) &= \int_{\mathcal{B}}\mathbf{C}_0(\mathbf{x},\mathbf{q})\, dV_{\mathbf{q}} = \mathbf{G} \\ \boldsymbol{\lambda}(\mathbf{x}) &= -\lim_{\epsilon\to 0}\int_{\mathcal{S}_\epsilon}\mathbf{C}_0(\mathbf{x},\mathbf{q})\, dV_{\mathbf{q}} = \mathbf{G} \\ \mathbf{C}(\mathbf{x},\mathbf{q}) &= \mathbf{C}_0(\mathbf{x},\mathbf{q}) + \boldsymbol{\lambda}(\mathbf{x})\Delta(\mathbf{q}-\mathbf{x}) = \boldsymbol{\gamma}(\mathbf{q}-\mathbf{x}) + \boldsymbol{\gamma}(\mathbf{x}-\mathbf{q}) \\ \mathbf{P}(\mathbf{x}) &= \mathbf{P}_0(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x}) = 2\mathbf{G}. \end{aligned} \tag{44}$$

From these expressions and (33), the equation of motion for this body is

$$\begin{aligned} \rho\ddot{\mathbf{u}}(\mathbf{x},t) &= \int_{\mathcal{B}}\big(\boldsymbol{\gamma}(\mathbf{q}-\mathbf{x}) + \boldsymbol{\gamma}(\mathbf{x}-\mathbf{q})\big)\mathbf{u}(\mathbf{q},t)\, dV_{\mathbf{q}} - 2\mathbf{G}\mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) \\ &= \int_{\mathcal{B}}\big(\boldsymbol{\gamma}(\mathbf{q}-\mathbf{x}) + \boldsymbol{\gamma}(\mathbf{x}-\mathbf{q})\big)(\mathbf{u}(\mathbf{q},t) - \mathbf{u}(\mathbf{x},t))\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x},t) \end{aligned}$$

$$= \int_{\mathcal{B}} c(|\mathbf{q}-\mathbf{x}|)(\mathbf{M}\otimes\mathbf{M})(\mathbf{u}(\mathbf{q},t)-\mathbf{u}(\mathbf{x},t))\,dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x},t)$$

where the *micromodulus* function is defined by

$$c(\boldsymbol{\xi}) = c(-\boldsymbol{\xi}) = \psi''(0,\boldsymbol{\xi}) + \psi''(0,-\boldsymbol{\xi})$$

and

$$\mathbf{M} = \frac{(\mathbf{q}-\mathbf{x})\otimes(\mathbf{q}-\mathbf{x})}{|\mathbf{q}-\mathbf{x}|^2}.$$

The same result could also be obtained using the linearization method given in [1] for the bond-based peridynamic theory with a pairwise bond force given by

$$\mathbf{f}(\mathbf{q},\mathbf{x}) = c(\boldsymbol{\xi})(\mathbf{M}\otimes\mathbf{M})\boldsymbol{\eta}$$

where

$$\boldsymbol{\eta} = \mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{x}), \qquad \boldsymbol{\xi} = \mathbf{q} - \mathbf{x}.$$

5.2 Example 2: Linear Fluid

A peridynamic inviscid fluid (see (100) of [1]) may be described in the following form:

$$\hat{W}(\underline{\mathbf{Y}}) = \frac{k\vartheta^2}{2} \tag{45}$$

where k is the usual bulk modulus and ϑ is a nonlocal dilatation defined by

$$\vartheta = \frac{3}{m}\int_{\mathcal{H}} \omega(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|\underline{e}\langle\boldsymbol{\xi}\rangle\,dV_{\boldsymbol{\xi}}, \quad m = \int_{\mathcal{H}} \omega(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2\,dV_{\boldsymbol{\xi}}. \tag{46}$$

Here, ω is a weighting function, $\underline{e}$ is defined in (40), and m is a normalization factor. As shown in [1], the corresponding force state is given by

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \frac{3k\vartheta}{m}\omega(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|\mathbf{M}, \quad \mathbf{M} = \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}. \tag{47}$$

This model for a fluid is constitutively linear in the sense that all the bond forces are proportional to ϑ. To evaluate $\underline{\mathbb{K}}$ for $\underline{\mathbf{Y}}^0 = \underline{\mathbf{X}}$, *i.e.,* for small displacements applied to the reference configuration, observe from (46) that

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \left(\frac{3k}{m}\omega(|\boldsymbol{\xi}|)\boldsymbol{\xi}\right)\left(\frac{3}{m}\int_{\mathcal{H}} \omega(|\boldsymbol{\zeta}|)\boldsymbol{\zeta}\cdot\underline{\mathbf{U}}\langle\boldsymbol{\zeta}\rangle\,dV_{\boldsymbol{\zeta}}\right)$$

hence, using (5) and (22),

$$\underline{\mathbb{K}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle = \frac{9k}{m^2}\omega(|\boldsymbol{\xi}|)\omega(|\boldsymbol{\zeta}|)\boldsymbol{\xi}\otimes\boldsymbol{\zeta}. \tag{48}$$

From the above definitions and (32)–(36), one finds for a homogeneous body at points far from the boundaries,

$$\mathbf{C}_0(\mathbf{x},\mathbf{q}) = \mathbf{C}(\mathbf{x},\mathbf{q}) \tag{49}$$

$$= \frac{-9k}{m^2} \int_{\mathcal{B}} \omega(|\mathbf{p}-\mathbf{x}|)\omega(|\mathbf{p}-\mathbf{q}|)(\mathbf{p}-\mathbf{x}) \otimes (\mathbf{p}-\mathbf{q})\, dV_{\mathbf{p}}, \tag{50}$$

$$\mathbf{P}_0(\mathbf{x}) = \mathbf{P}(\mathbf{x}) = \boldsymbol{\lambda}(\mathbf{x}) = \mathbf{0},$$

and the equation of motion for this body is

$$\rho\ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{\mathcal{B}} \mathbf{C}(\mathbf{x}, \mathbf{q})\mathbf{u}(\mathbf{q})\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}, t).$$

5.3 Example 3: Linear Isotropic Solid

A model for an elastic, peridynamic, constitutively linear, isotropic solid is obtained by adding a term to the strain energy density for a linear fluid (45) that involves the deviatoric part of the deformation state. (See (105) of [1] for a discussion of this term.) The resulting strain energy density function may be written as

$$\hat{W}(\underline{\mathbf{Y}}) = \frac{k\vartheta^2}{2} + \frac{\alpha}{2} \int_{\mathcal{H}} \omega(|\boldsymbol{\xi}|) \left(\underline{e}\langle\boldsymbol{\xi}\rangle - \frac{\vartheta|\boldsymbol{\xi}|}{3} \right)^2 dV_{\boldsymbol{\xi}} \tag{51}$$

where k, ϑ, and $\underline{e}$ are as defined in the previous two examples, and α is a constant. The deviatoric part of the extension state, which appears above in the term $\underline{e}\langle\boldsymbol{\xi}\rangle - \vartheta|\boldsymbol{\xi}|/3$, represents the bond extension after subtracting off an isotropic expansion of the family $\mathcal{H}$ with dilatation ϑ. This deviatoric part includes not only shear, but also any deformation of $\mathcal{H}$ other than isotropic expansion. It is shown in [1] that $\alpha = 15\mu/m$, where μ is the usual shear modulus.

Using (46), (51) can be rewritten as

$$\hat{W}(\underline{\mathbf{Y}}) = \frac{1}{2}\left(k - \frac{\alpha m}{9}\right)\vartheta^2 + \frac{\alpha}{2} \int_{\mathcal{H}} \omega(|\boldsymbol{\xi}|)\underline{e}^2\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}}.$$

The first term on the right hand side of this equation is the same as for the fluid in Example 2, but with a different constant. The second term is a special case of the bond-based material energy in Example 1 with

$$\psi(e, \boldsymbol{\xi}) = \frac{\alpha}{2}\omega(|\boldsymbol{\xi}|)e^2.$$

Therefore, from (42) and (47),

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \left(\frac{3k}{m} - \frac{\alpha}{3}\right)\omega(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|\vartheta\mathbf{M} + \alpha\omega(|\boldsymbol{\xi}|)\underline{e}\langle\boldsymbol{\xi}\rangle\mathbf{M}, \quad \mathbf{M} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}.$$

Similarly, using (43) and (48),

$$\underline{\mathbb{K}}\langle\boldsymbol{\xi}, \boldsymbol{\zeta}\rangle = \left(\frac{9k}{m^2} - \frac{\alpha}{m}\right)\omega(|\boldsymbol{\xi}|)\omega(|\boldsymbol{\zeta}|)\boldsymbol{\xi} \otimes \boldsymbol{\zeta} + \boldsymbol{\gamma}(\boldsymbol{\xi})\Delta(\boldsymbol{\zeta} - \boldsymbol{\xi})$$

where

$$\boldsymbol{\gamma}(\boldsymbol{\xi}) = \alpha\omega(|\boldsymbol{\xi}|)(\mathbf{M} \otimes \mathbf{M}).$$

Also, from (44) and (50),

$$\mathbf{C}(\mathbf{x},\mathbf{q}) = \left(\frac{\alpha}{m} - \frac{9k}{m^2}\right)\int_{\mathcal{B}} \omega(|\mathbf{p}-\mathbf{x}|)\omega(|\mathbf{p}-\mathbf{q}|)(\mathbf{p}-\mathbf{x})\otimes(\mathbf{p}-\mathbf{q})\,dV_{\mathbf{p}} + 2\boldsymbol{\gamma}(\mathbf{q}-\mathbf{x}),$$

$$\mathbf{P}(\mathbf{x}) = 2\int_{\mathcal{H}} \boldsymbol{\gamma}(\boldsymbol{\xi})\,dV_{\boldsymbol{\xi}}.$$

5.4 Example 4: Single-Bond Material with Constant Force Magnitudes

This is a special case of a bond-based material in which each point has only one bond $\boldsymbol{\tau}$ that is capable of sustaining force, and the magnitude of the force is constant:

$$\hat{W}(\underline{\mathbf{Y}}) = f_0 \underline{e}\langle\boldsymbol{\tau}\rangle$$

where f_0 is a constant. Writing this in the form

$$\hat{W}(\underline{\mathbf{Y}}) = f_0 \int_{\mathcal{H}} \underline{e}\langle\boldsymbol{\xi}\rangle \Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\,dV_{\boldsymbol{\xi}}$$

and using (41) leads to

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = f_0\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\mathbf{M}, \quad \mathbf{M} = \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}. \tag{52}$$

This type of model could reasonably represent a material composed of unidirectional fibers with no adhesion or friction between them, in which the axial forces in the fibers are insensitive to the stretch. We carry out the linearization process near the reference configuration, allowing for nonzero f_0. To do this, first observe that the only term in the expression for $\underline{\mathbf{T}}$ that depends on $\underline{\mathbf{U}}$ is $\mathbf{M}$. Differentiating $\mathbf{M}$, and evaluating the result in the reference configuration, in which $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi}$, leads to

$$d\mathbf{M} = \frac{1}{|\boldsymbol{\xi}|}(\mathbf{1}-\mathbf{M}\otimes\mathbf{M})d\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle.$$

Since $\underline{\mathbf{U}}$ is a small displacement state, we set $d\underline{\mathbf{Y}} = \underline{\mathbf{U}}$ in this equation for purposes of linearization and combine the result with (52), resulting in

$$\begin{aligned}\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle &= f_0\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})(\mathbf{M}+d\mathbf{M})\\ &= f_0\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\mathbf{M} + f_0\mathbf{N}\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\underline{\mathbf{U}}\langle\boldsymbol{\xi}\rangle\\ &= f_0\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\mathbf{M} + f_0\mathbf{N}\int_{\mathcal{H}}\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\Delta(\boldsymbol{\zeta}-\boldsymbol{\tau})\underline{\mathbf{U}}\langle\boldsymbol{\zeta}\rangle\,dV_{\boldsymbol{\zeta}}\end{aligned}$$

where $\mathbf{N}$ is a constant tensor given by

$$\mathbf{N} = \frac{1}{|\boldsymbol{\tau}|}\left(\mathbf{1} - \frac{\boldsymbol{\tau}\otimes\boldsymbol{\tau}}{|\boldsymbol{\tau}|^2}\right). \tag{53}$$

Hence we can write

$$\underline{\mathbf{T}} = \underline{\mathbf{T}}^0 + \underline{\mathbb{K}} \bullet \underline{\mathbf{U}}$$

where

$$\underline{\mathbf{T}}^0\langle\boldsymbol{\xi}\rangle = f_0\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\frac{\boldsymbol{\tau}}{|\boldsymbol{\tau}|}$$

and

$$\underline{\mathbb{K}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle = f_0\mathbf{N}\Delta(\boldsymbol{\xi}-\boldsymbol{\tau})\Delta(\boldsymbol{\zeta}-\boldsymbol{\tau}).$$

Proceeding as in Example 1 to find the linearized equation of motion,

$$\mathbf{C}_0(\mathbf{x},\mathbf{q}) = f_0\mathbf{N}\big(\Delta(\mathbf{q}-\mathbf{x}-\boldsymbol{\tau})+\Delta(\mathbf{x}-\mathbf{q}-\boldsymbol{\tau})-\Delta(\mathbf{q}-\mathbf{x})\big),$$

$$\mathbf{P}_0(\mathbf{x}) = f_0\mathbf{N}.$$

Since this $\mathbf{C}_0$ contains a term of the form $\Delta(\mathbf{q}-\mathbf{x})$, we can combine this term with $\mathbf{P}_0$ as indicated in (33)–(36) to obtain

$$\boldsymbol{\lambda}(\mathbf{x}) = f_0\mathbf{N}, \qquad \mathbf{P}(\mathbf{x}) = 2f_0\mathbf{N},$$

$$\mathbf{C}(\mathbf{x},\mathbf{q}) = f_0\mathbf{N}\big(\Delta(\mathbf{q}-\mathbf{x}-\boldsymbol{\tau})+\Delta(\mathbf{x}-\mathbf{q}-\boldsymbol{\tau})\big).$$

The linearized equation of motion is therefore

$$\begin{aligned}\rho\ddot{\mathbf{u}}(\mathbf{x},t) = {} & f_0\mathbf{N}\int_{\mathcal{B}}\big(\Delta(\mathbf{q}-\mathbf{x}-\boldsymbol{\tau})+\Delta(\mathbf{x}-\mathbf{q}-\boldsymbol{\tau})\big)\mathbf{u}(\mathbf{q},t)\,dV_{\mathbf{q}} \\ & -2f_0\mathbf{N}\mathbf{u}(\mathbf{x},t)+\mathbf{b}(\mathbf{x},t)\end{aligned}$$

or

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = f_0\mathbf{N}\big(\mathbf{u}(\mathbf{x}+\boldsymbol{\tau},t)-2\mathbf{u}(\mathbf{x},t)+\mathbf{u}(\mathbf{x}-\boldsymbol{\tau},t)\big)+\mathbf{b}(\mathbf{x},t). \tag{54}$$

Because of the form of $\mathbf{N}$ given in (53), it follows that $\mathbf{N}\boldsymbol{\tau}=\mathbf{0}$, which implies that displacements parallel to the bond direction $\boldsymbol{\tau}$ result in no change in force in the bond. Because of this, the linearized equation of motion (54) always predicts zero acceleration parallel to the bonds.

However, the linearized equation of motion predicts nonzero *transverse* accelerations. These transverse accelerations give rise to shear waves propagating parallel to $\boldsymbol{\tau}$. To see this, assume displacements of the form $\mathbf{u}=\mathbf{a}\exp i(\kappa\mathbf{n}\cdot\mathbf{x}-\omega t)$ where ω is the angular wave frequency, κ is the wave number, the propagation direction is $\mathbf{n}=\boldsymbol{\tau}/|\boldsymbol{\tau}|$, and $\mathbf{a}\perp\mathbf{n}$ is the amplitude vector. Using this expression in (54) with $\mathbf{b}=\mathbf{0}$, one arrives at the following dispersion relation:

$$\omega^2 = \frac{2f_0}{\rho|\boldsymbol{\tau}|}\big(1-\cos(|\boldsymbol{\tau}|\kappa)\big). \tag{55}$$

This is the similar to the dispersion relation for an infinite, one-dimensional spring-mass lattice, but in this case the spring constant is determined by the pre-existing bond force density f_0. If this force density is tensile ($f_0>0$), then stable transverse waves can exist. The waves, although dispersive, are similar to transverse waves in a stretched rubber band. However, if the force density is compressive ($f_0<0$), then the right hand side of (55) can be negative, and transverse oscillations are therefore unstable. This instability is reminiscent of the microbuckling mechanism in fiber-reinforced composite materials, which causes failure in compression due to the transverse deformation of the fibers [14, 15].

For simplicity, in this example, linearization was performed in the reference configuration. Similar results would be obtained in any other configuration, but the waves would be transverse to the bond orientation where linearization occurs.

This example illustrates the role of finite bond forces in the reference configuration, if they are present, and how they interact with a small superposed displacement field. The linearized bond forces transverse to $\boldsymbol{\tau}$ arise because of a purely geometrical effect. The rotation of the bonds containing a force density f_0 has a first order effect on the bond forces when the small displacement state $\underline{\mathbf{U}}$ is applied. In Sect. 6, we will return briefly to the significance of this coupling in modeling atomic scale systems.

6 Discussion

The linearized state-based theory developed in this paper has basically the same structure as the linearized bond-based theory [3]. The equation of motion (33) is formally the same. However, as noted in the discussion before (37), $\mathbf{C}$ has additional restrictions in the bond-based theory. Also, in the linearized state-based version, the effective maximum interaction distance between points is 2δ, where δ is the horizon in the original state-based constitutive model that was linearized. (See Fig. 2.)

The linearization process described above preserves the ability of the nonlinear peridynamic model to treat discontinuities in displacement, such as cracks, according to the same equations as continuous deformations. This treatment of discontinuities is possible because there is nothing in the notion of a small deformation in (19), (20) that requires smoothness of the displacement field.

The peridynamic theory has been developed in such a way as to emphasize the parallels between it and the standard theory. Table 1 lists some of the quantities, operations, and equations that are analogous between the two theories.

The linear theory described here can serve as the basis for an incremental form of peridynamic constitutive models. The only change is that the modulus state becomes time-dependent, thus

$$\dot{\underline{\mathbf{T}}} = \underline{\mathbb{K}}[\mathbf{x}, t] \bullet \dot{\underline{\mathbf{Y}}}.$$

In such an incremental version, if the material is elastic, Lemma A.4 continues to apply, hence the time-dependent modulus state is self-adjoint. (Conversely, if it is self-adjoint, then the material is elastic.)

Certain approaches to nonlocal elasticity take linear integro-differential equations similar to (33), or their equivalent variational form, as a starting point [11–13]. These theories do not include the underlying structure presented here for material models based on peridynamic states. This additional structure tends to make the entire model more complete and useful by providing a way to determine the material-dependent functions $\mathbf{C}$ and $\mathbf{P}$ in the linear integro-differential equation (33). For example, if we start with (33) and ask how to represent a fluid within this model, it is not obvious how to choose $\mathbf{C}$ and $\mathbf{P}$. In the peridynamic approach, these quantities emerge in a natural way as shown in Example 2 above. Also, as shown in the present paper, the linear peridynamic theory is derivable as a special case of a more general, nonlinear theory, by introducing only the assumption of small deformation.

The linearized peridynamic approach also allows for finite internal forces $\underline{\mathbf{T}}^0$ that may be present initially when linearization is applied. These forces are important in the small scale modeling of many materials. In an unstressed crystal, for example, there are strong forces between the individual atoms; the stress tensor is zero if the compressive and tensile forces

Table 1 Analogies between the standard and peridynamic theories. The symbol ∇ denotes the Fréchet derivative. Linearization takes place following a large equilibrated deformation with stress tensor field $\boldsymbol{\sigma}^0$ or force state field $\underline{\mathbf{T}}^0$

Quantity or relation	Standard	Peridynamic
Required smoothness	$\mathbf{y} \in \mathcal{C}^2(\mathcal{B})$ (strong) $\mathbf{y} \in \mathcal{C}^1_{\text{pw}}(\mathcal{B}) \cap \mathcal{C}(\mathcal{B})$ (weak)	$\mathbf{y}$ integrable on $\mathcal{B}$
Fundamental description of internal forces	Piola stress tensor $\boldsymbol{\sigma}$	Force state $\underline{\mathbf{T}}$
What internal forces depend on (nonlinear theory)	$\mathbf{F} = \text{grad}\,\mathbf{y}$	$\underline{\mathbf{Y}}\langle\mathbf{q}-\mathbf{x}\rangle = \mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{x})$
What internal forces depend on (linearized)	$\mathbf{H} = \text{grad}\,\mathbf{u}$	$\underline{\mathbf{U}}\langle\mathbf{q}-\mathbf{x}\rangle = \mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{x})$
Constitutive model (nonlinear theory)	$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{F})$	$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})$
Elastic constitutive model (nonlinear theory)	$\hat{\boldsymbol{\sigma}} = \partial\hat{W}/\partial\mathbf{F}$	$\hat{\underline{\mathbf{T}}} = \nabla\hat{W}$
Rate of change of strain energy density	$\dot{W} = \boldsymbol{\sigma}\cdot\dot{\mathbf{F}}$	$\dot{W} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}}$
Elastic constitutive model (linearized)	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \mathsf{C}\mathbf{H}$, $\mathsf{C} = \partial^2\hat{W}/\partial\mathbf{F}^2$	$\underline{\mathbf{T}} = \underline{\mathbf{T}}^0 + \underline{\mathbb{K}} \bullet \underline{\mathbf{U}}$, $\quad \underline{\mathbb{K}} = \nabla\nabla\hat{W}$
Condition for linearization	$\|\mathbf{H}\| \ll 1$	$\sup_{\mathbf{q}\in\mathcal{H}} \|\mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{x})\| \ll \delta$
Equation of motion (nonlinear theory)	$\rho\ddot{\mathbf{y}} = \text{div}\,\boldsymbol{\sigma} + \hat{\mathbf{b}}$	$\rho\ddot{\mathbf{y}} = \int \big(\underline{\mathbf{T}}[\mathbf{x}]\langle\mathbf{q}-\mathbf{x}\rangle$ $-\underline{\mathbf{T}}[\mathbf{q}]\langle\mathbf{x}-\mathbf{q}\rangle\big)dV_{\mathbf{q}} + \hat{\mathbf{b}}$
Equation of motion (linearized)	$\rho\ddot{\mathbf{u}} = \text{div}(\mathsf{C}\mathbf{H}) + \mathbf{b}$	$\rho\ddot{\mathbf{u}} = \int \mathbf{C}(\mathbf{x},\mathbf{q})\mathbf{u}(\mathbf{q})dV_{\mathbf{q}}$ $-\mathbf{P}\mathbf{u} + \mathbf{b}$
Balance of angular momentum (nonlinear theory)	$\boldsymbol{\sigma}\mathbf{F}^T - \mathbf{F}\boldsymbol{\sigma}^T = \mathbf{0}$	$\int \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle dV_{\boldsymbol{\xi}} = \mathbf{0}$
Condition for a linear material to be elastic	$\mathsf{C} = \mathsf{C}^T$	$\underline{\mathbb{K}} = \underline{\mathbb{K}}^\dagger$

acting through any surface cancel each other out. As shown in Example 4 above, these finite nonlocal forces in the reference configuration interact with bond rotations when a small deformation is applied, creating a first order effect on bond forces. Linear nonlocal theories that do not include terms like $\mathbf{T}^0$ therefore may be neglecting a potentially significant effect.

The peridynamic approach to modeling a continuous body involves summing up the forces between points separated from each other by finite distances. The similarity between this and molecular dynamics raises the question of whether an elastic peridynamic material model, *i.e.,* the strain energy density function $\hat{W}$, could be obtained directly from an interatomic potential. Preliminary work on pair potentials suggests that this is possible if suitable homogenization and rescaling techniques are applied [16]. Extension of this approach to multibody potentials, including fundamental improvements in homogenization and rescaling methods, is currently an active area of study.

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Appendix: Properties of Second Fréchet Derivatives

A.1 Differentiation by a Scalar Parameter

The following result follows immediately from (4):

Lemma A.1 *Let $\Psi(\cdot):\mathcal{V}\to\mathbb{R}$, and let Ψ be Fréchet differentiable. For fixed $\underline{\mathbf{A}}, \underline{\mathbf{a}}\in\mathcal{V}$, define a function $\psi(\cdot):\mathbb{R}\to\mathbb{R}$ by*

$$\psi(\alpha)=\Psi(\underline{\mathbf{A}}+\alpha\underline{\mathbf{a}}).$$

Then

$$\frac{d\psi}{d\alpha}(\alpha)=\nabla\Psi(\underline{\mathbf{A}})\bullet\underline{\mathbf{a}}.$$

Lemma A.1 remains true if Ψ and ψ are state-valued.

A.2 Exchange of Mixed Fréchet Derivatives

Let $\Phi(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{R}$. Denote its Fréchet derivatives with respect to its first and second arguments by ∇_1 and ∇_2 respectively, each holding the other argument fixed. Let the second Fréchet derivatives of Φ be the double states denoted

$$\nabla_1\nabla_1\Phi=\nabla_1(\nabla_1\Phi),\quad \nabla_1\nabla_2\Phi=\nabla_1(\nabla_2\Phi),\quad \ldots.$$

Lemma A.2 *Let $\Phi(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ be twice continuously Fréchet differentiable in both its arguments. Then*

$$\nabla_1\nabla_2\Phi=(\nabla_2\nabla_1\Phi)^\dagger,\qquad \nabla_2\nabla_1\Phi=(\nabla_1\nabla_2\Phi)^\dagger\quad \text{on } \mathcal{V}\times\mathcal{V}. \tag{56}$$

Proof For fixed $\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{a}}, \underline{\mathbf{b}}\in\mathcal{V}$, let

$$\phi(\alpha,\beta)=\Phi(\underline{\mathbf{A}}+\alpha\underline{\mathbf{a}}, \underline{\mathbf{B}}+\beta\underline{\mathbf{b}}),$$

and observe that ϕ is twice continuously differentiable in both variables. Then from an obvious extension of Lemma A.1 to functions of two variables,

$$\frac{\partial\phi}{\partial\alpha}(\alpha,\beta)=\nabla_1\Phi\bullet\underline{\mathbf{a}},\qquad \frac{\partial\phi}{\partial\beta}(\alpha,\beta)=\nabla_2\Phi\bullet\underline{\mathbf{b}}. \tag{57}$$

Using (57) and differentiating again,

$$\frac{\partial}{\partial \beta}\left(\frac{\partial \phi}{\partial \alpha}\right) = (\nabla_2 \nabla_1 \Phi \bullet \underline{\mathbf{b}}) \bullet \underline{\mathbf{a}} = \underline{\mathbf{a}} \bullet \nabla_2 \nabla_1 \Phi \bullet \underline{\mathbf{b}} \tag{58}$$

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial \phi}{\partial \beta}\right) = (\nabla_1 \nabla_2 \Phi \bullet \underline{\mathbf{a}}) \bullet \underline{\mathbf{b}} = \underline{\mathbf{b}} \bullet \nabla_1 \nabla_2 \Phi \bullet \underline{\mathbf{a}}. \tag{59}$$

Because of the interchangeability of mixed partial derivatives of ϕ, (58) and (59) are equal, so

$$\underline{\mathbf{a}} \bullet \nabla_2 \nabla_1 \Phi \bullet \underline{\mathbf{b}} = \underline{\mathbf{b}} \bullet \nabla_1 \nabla_2 \Phi \bullet \underline{\mathbf{a}}.$$

From this and (3),

$$\underline{\mathbf{a}} \bullet \left(\nabla_2 \nabla_1 \Phi - (\nabla_1 \nabla_2 \Phi)^{\dagger}\right) \bullet \underline{\mathbf{b}} = 0.$$

Since this must hold for every choice of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, (56) follows. □

A.3 Self-adjointness of Second Fréchet Derivatives

Lemma A.3 *Let $\Psi(\cdot) : \mathcal{V} \to \mathbb{R}$. If Ψ is twice continuously Fréchet differentiable, then*

$$(\nabla\nabla\Psi)^{\dagger} = \nabla\nabla\Psi \quad \textit{on } \mathcal{V}. \tag{60}$$

Proof Define a function $\Gamma(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ by

$$\Gamma(\underline{\mathbf{a}}, \underline{\mathbf{b}}) = \Psi\left(\frac{\underline{\mathbf{a}} + \underline{\mathbf{b}}}{2}\right) \quad \forall \underline{\mathbf{a}}, \underline{\mathbf{b}} \in \mathcal{V}.$$

Evaluating the second Fréchet derivatives of Γ and setting $\underline{\mathbf{b}} = \underline{\mathbf{a}}$,

$$\nabla_1 \nabla_2 \Gamma(\underline{\mathbf{a}}, \underline{\mathbf{a}}) = \nabla_2 \nabla_1 \Gamma(\underline{\mathbf{a}}, \underline{\mathbf{a}}) = \frac{\nabla\nabla\Psi(\underline{\mathbf{a}})}{4}.$$

Therefore, by Lemma A.2, (60) follows. □

A.4 Poincaré's Theorem for States

Lemma A.4 *Let $\underline{\mathbf{S}}(\cdot) : \mathcal{V} \to \mathcal{V}$ be continuously Fréchet differentiable. A necessary and sufficient condition for there to exist a scalar valued function $\Psi(\cdot) : \mathcal{V} \to \mathbb{R}$ such that*

$$\underline{\mathbf{S}} = \nabla\Psi \quad \textit{on } \mathcal{V} \tag{61}$$

is that

$$\nabla\underline{\mathbf{S}} = (\nabla\underline{\mathbf{S}})^{\dagger} \quad \textit{on } \mathcal{V}. \tag{62}$$

Proof (i) *Necessity.* If (61) holds, then (62) follows immediately from Lemma A.3.

(ii) *Sufficiency.* Suppose (62) holds. Define $\Psi(\cdot)$ by

$$\Psi(\underline{\mathbf{A}}) = \int_0^1 \underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet \underline{\mathbf{A}}\, dp$$

for any $\underline{\mathbf{A}} \in \mathcal{V}$. We will evaluate directly the Fréchet derivative of this Ψ. To do this, use (4) to obtain, for any $\underline{\mathbf{a}} \in \mathcal{V}$,

$$\begin{aligned}
\Psi(\underline{\mathbf{A}}+\underline{\mathbf{a}}) - \Psi(\underline{\mathbf{A}}) &= \int_0^1 \underline{\mathbf{S}}(p\underline{\mathbf{A}}+p\underline{\mathbf{a}}) \bullet (\underline{\mathbf{A}}+\underline{\mathbf{a}})\,dp - \int_0^1 \underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet \underline{\mathbf{A}}\,dp \\
&= \int_0^1 \Big(\underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet (\underline{\mathbf{A}}+\underline{\mathbf{a}}) + (\nabla\underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet p\underline{\mathbf{a}}) \bullet \underline{\mathbf{A}}\Big)\,dp \\
&\quad - \int_0^1 \underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet \underline{\mathbf{A}}\,dp + o(\|\underline{\mathbf{a}}\|) \\
&= \int_0^1 \Big(\underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet \underline{\mathbf{a}} + p\underline{\mathbf{A}} \bullet \nabla\underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet \underline{\mathbf{a}}\Big)\,dp + o(\|\underline{\mathbf{a}}\|) \\
&= \int_0^1 \Big(\underline{\mathbf{a}} \bullet \underline{\mathbf{S}}(p\underline{\mathbf{A}}) + p\underline{\mathbf{a}} \bullet (\nabla\underline{\mathbf{S}}(p\underline{\mathbf{A}}))^\dagger \bullet \underline{\mathbf{A}}\Big)\,dp + o(\|\underline{\mathbf{a}}\|).
\end{aligned} \tag{63}$$

Define a function $\underline{\mathbf{s}}(\cdot) : \mathbb{R} \to \mathcal{V}$ by $\underline{\mathbf{s}}(p) = \underline{\mathbf{S}}(p\underline{\mathbf{A}})$. Then by (4),

$$\frac{d\underline{\mathbf{s}}}{dp}(p) = \nabla\underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet \underline{\mathbf{A}}.$$

Using this, (62), and (63) leads to

$$\begin{aligned}
\Psi(\underline{\mathbf{A}}+\underline{\mathbf{a}}) - \Psi(\underline{\mathbf{A}}) &= \underline{\mathbf{a}} \bullet \int_0^1 \Big(\underline{\mathbf{S}}(p\underline{\mathbf{A}}) + p\nabla\underline{\mathbf{S}}(p\underline{\mathbf{A}}) \bullet \underline{\mathbf{A}}\Big)\,dp + o(\|\underline{\mathbf{a}}\|) \\
&= \underline{\mathbf{a}} \bullet \int_0^1 \Big(\underline{\mathbf{s}}(p) + p\frac{d\underline{\mathbf{s}}}{dp}(p)\Big)\,dp + o(\|\underline{\mathbf{a}}\|) \\
&= \underline{\mathbf{a}} \bullet \int_0^1 \frac{d}{dp}\Big(p\underline{\mathbf{s}}(p)\Big)\,dp + o(\|\underline{\mathbf{a}}\|) \\
&= \underline{\mathbf{a}} \bullet \Big[p\underline{\mathbf{s}}(p)\Big]_{p=0}^{p=1} + o(\|\underline{\mathbf{a}}\|) \\
&= \underline{\mathbf{a}} \bullet \Big[p\underline{\mathbf{S}}(p\underline{\mathbf{A}})\Big]_{p=0}^{p=1} + o(\|\underline{\mathbf{a}}\|) \\
&= \underline{\mathbf{a}} \bullet \underline{\mathbf{S}}(\underline{\mathbf{A}}) + o(\|\underline{\mathbf{a}}\|) \\
&= \underline{\mathbf{S}}(\underline{\mathbf{A}}) \bullet \underline{\mathbf{a}} + o(\|\underline{\mathbf{a}}\|).
\end{aligned}$$

Comparing this with (4), evidently

$$\underline{\mathbf{S}}(\underline{\mathbf{A}}) = \nabla\Psi(\underline{\mathbf{A}}),$$

establishing (61). □

References

1. Silling, S.A., Epton, M., Weckner, O., Xu, J., Askari, E.: Peridynamic states and constitutive modeling. J. Elast. **88**, 151–184 (2007)

2. Silling, S.A., Weckner, O., Askari, E., Bobaru, F.: Crack nucleation in a peridynamic solid. Int. J. Frac. (to appear)
3. Silling, S.A.: Reformulation of elasticity theory for discontinuities and long-range forces. J. Mech. Phys. Solids **48**, 175–209 (2000)
4. Silling, S.A., Zimmermann, M., Abeyaratne, R.: Deformation of a peridynamic bar. J. Elast. **73**, 173–190 (2003)
5. Weckner, O., Abeyaratne, R.: The effect of long-range forces on the dynamics of a bar. J. Mech. Phys. Solids **53**, 705–728 (2005)
6. Emmrich, E., Weckner, O.: Analysis and numerical approximation of an integro-differential equation modeling non-local effects in linear elasticity. Math. Mech. Solids **12**, 363–384 (2007)
7. Emmrich, E., Weckner, O.: On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of linear elasticity. Commun. Math. Sci. **5**, 851–864 (2007)
8. Alali, B.: Multiscale analysis of heterogeneous media for local and nonlocal continuum theories. Ph.D. thesis, Louisiana State University (2008)
9. Bobaru, F., Yang, M., Alves, L.F., Silling, S.A., Askari, E., Xu, J.: Convergence, adaptive refinement, and scaling in 1D peridynamics. Int. J. Numer. Methods Eng. **77**, 852–877 (2009)
10. Silling, S.A., Lehoucq, R.B.: Convergence of peridynamics to classical elasticity theory. J. Elast. **93**, 13–37 (2008)
11. Kunin, I.A.: Elastic Media with Microstructure I: One-Dimensional Models. Springer, Berlin (1982)
12. Kunin, I.A.: Elastic Media with Microstructure II: Three-Dimensional Models. Springer, Berlin (1983)
13. Rogula, D.: Nonlocal Theory of Material Media. Springer, Berlin (1982). pp. 137–149 and 243–278
14. Rosen, B.W.: Mechanics of composite strengthening. In: Fiber Composite Materials, pp. 37–75. American Society of Metals, Metals Park (1964)
15. Drapier, S., Grandidier, J.-C., Jochum, C., Potier-Ferry, M.: Structural plastic microbuckling and compressive strength of long-fibre composite materials. In: Maugin, G.A., et al. (eds.) Continuum Thermomechanics, pp. 125–136. Kluwer, New York (2000)
16. Lehoucq, R.B., Silling, S.A.: Statistical coarse-graining of molecular dynamics into peridynamics. Sandia National Laboratories report, SAND2007-6410 (2007)